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On soft topological spaces

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ABSTRACT

In the present paper we introduce soft topological spaces which are defined over an initial universe with a fixed set of parameters. The notions of soft open sets, soft closed sets, soft closure, soft interior points, soft neighborhood of a point and soft separation axioms are introduced and their basic properties are investigated. It is shown that a soft topological space gives a parametrized family of topological spaces. Furthermore, with the help of an example it is established that the converse does not hold. The soft subspaces of a soft topological space are defined and inherent concepts as well as the characterization of soft open and soft closed sets in soft subspaces are investigated. Finally, soft T_i-*spaces* and notions of soft normal and soft regular spaces are discussed in detail. A sufficient condition for a soft topological space to be a soft T_1-*space* is also presented.

1. Introduction

The real world is too complex for our immediate and direct understanding. We create "models" of reality that are simplifications of aspects of the real word. Unfortunately these mathematical models are too complicated and we cannot find the exact solutions. The uncertainty of data while modeling the problems in engineering, physics, computer sciences, economics, social sciences, medical sciences and many other diverse fields makes it unsuccessful to use the traditional classical methods. These may be due to the uncertainties of natural environmental phenomena, of human knowledge about the real world or to the limitations of the means used to measure objects. For example, vagueness or uncertainty in the boundary between states or between urban and rural areas or the exact growth rate of population in a country's rural area or making decisions in a machine based environment using database information. Thus classical set theory, which is based on the crisp and exact case may not be fully suitable for handling such problems of uncertainty.

There are several theories, for example, theory of fuzzy sets [1], theory of intuitionistic fuzzy sets [2], theory of vague sets, theory of interval mathematics [3,4] and theory of rough sets [5]. These can be considered as tools for dealing with uncertainties but all these theories have their own difficulties. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory as it was mentioned by Molodtsov in [6]. He initiated the concept of soft set theory as a new mathematical tool which is free from the problems mentioned above. In his paper [6], he presented the fundamental results of the new theory and successfully applied it to several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, theory of probability etc. A soft set is a collection of approximate descriptions of an object. He also showed how soft set theory is free from the parametrization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory and game theory. Soft systems provide a very general framework with the involvement of parameters. Research works on soft set theory and its applications in various fields are progressing rapidly.

Maji et al. [7,8] presented an application of soft sets in decision making problems that is based on the reduction of parameters to keep the optimal choice objects. Chen [9] presented a new definition of soft set parametrization reduction

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and a comparison of it with attribute reduction in rough set theory. Pei and Miao [10] showed that soft sets are a class of special information systems. Kong et al. [11] introduced the notion of normal parameter reduction of soft sets and its use to investigate the problem of sub-optimal choice and added a parameter set in soft sets. Zou and Xiao [12] discussed the soft data analysis approach. The application of soft set theory in algebraic structures was introduced by Aktaş and Çağman [13]. They discussed the notion of soft groups and derived some basic properties. They also showed that soft groups extended the concept of fuzzy groups. Jun [14,15] investigated soft BCK/BCI-algebras and its application in ideal theory. Feng et al. [16] worked on soft semirings, soft ideals and idealistic soft semirings. Ali et al. [17] and Shabir and Irfan Ali [17,18] studied soft semigroups and soft ideals over a semigroup which characterize generalized fuzzy ideals and fuzzy ideals with thresholds of a semigroup.

The main purpose of this paper is to introduce soft topological spaces which are defined over an initial universe with a fixed set of parameters. First we give some basic ideas about soft sets and the results already studied. Then we discuss some basic properties of soft topological spaces and define soft open and closed sets. The soft closure of a soft set is defined which is, in fact, a generalization of closure of a set in a broader sense. The newly introduced concept of parameters comes into play with the collection of parametrized topologies on the initial universe. Corresponding to each parameter, we get a topological space and this makes the involvement of parameters more significant. We can say that a soft topological space gives a parametrized family of topologies on the initial universe but the converse is not true i.e. we cannot construct a soft topological space if we are given some topologies for each parameter and this is shown in detail with the help of examples in this paper. Consequently we can say that the soft topological spaces are more comprehensive and generalized than the classical topological spaces. During the process of theory development we also see that the properties of parametrized topologies correspond to that of soft spaces in some particular situations. Finally soft separation axioms for soft topological spaces are defined and some interesting results are derived which may be of value for further research. Although most of the results, discussed in this paper, are very basic and provide an introductory platform but potentially useful research in theoretical as well as applicable directions can be made. One possible inspirational thought is of modeling topological relations between spatial objects. Mathematically, point set topology can be applied as a fundamental tool for modeling crisp spatial objects in GIS. Spatial and non-spatial information about entities is represented by attributes. The models developed so far, do not consider attribute aspects of spatial objects so the soft topology may be a handy tool for this purpose. The resemblance of soft sets with that of a query from databases may be a useful side for the modeling process.

2. Soft sets

Definition 1 ([6]). Let U be an initial universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$.

In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

Definition 2 ([8]). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) for all $e \in A$, $F(e)$ and $G(e)$ are identical approximations.

We write $(F, A)\widetilde{\subset}(G, B)$.

(F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\widetilde{\supset}(G, B)$.

Definition 3 ([8]). Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 4 ([8]). Let $E = \{e_1, e_2, \ldots, e_n\}$ be a set of parameters. The *NOT* set of E denoted by $\rceil E$ is defined by $\rceil E = \{\rceil e_1, \rceil e_2, \ldots, \rceil e_n\}$ where, $\rceil e_i = not\ e_i$ for all i.

The following results are obvious.

Proposition 1 ([8]).

(1) $\rceil(\rceil A) = A$;
(2) $\rceil(A \cup B) = \rceil A \cup \rceil B$;
(3) $\rceil(A \cap B) = \rceil A \cap \rceil B$.

Definition 5 ([8]). The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A)$ where, $F^c : \rceil A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U \setminus F(\rceil\alpha)$, for all $\alpha \in \rceil A$.

Let us call F^c to be the soft complement function of F. Clearly $(F^c)^c$ is the same as F and $((F, A)^c)^c = (F, A)$.

Definition 6 ([8]). A soft set (F, A) over U is said to be a *NULL* soft set denoted by Φ if for all $\varepsilon \in A$, $F(\varepsilon) = \emptyset$ (null set).

Definition 7 ([8]). A soft set (F, A) over U is said to be an *absolute* soft set denoted by $\widetilde{A}$ if for all $\varepsilon \in A$, $F(\varepsilon) = U$.

Clearly $\widetilde{A}^c = \Phi$ and $\Phi^c = \widetilde{A}$.

Definition 8 ([8]). If (F, A) and (G, B) are two soft sets then "(F, A) *AND* (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H((\alpha, \beta)) = F(\alpha) \cap G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 9 ([8]). If (F, A) and (G, B) are two soft sets then "(F, A) *OR* (G, B)" denoted by $(F, A) \vee (G, B)$ is defined by $(F, A) \vee (G, B) = (O, A \times B)$ where, $O((\alpha, \beta)) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

It is shown in [8] that the following De Morgan's type of results are true.

Proposition 2 ([17]).
(1) $((F, A) \vee (G, B))^c = (F, A)^c \wedge (G, B)^c$
(2) $((F, A) \wedge (G, B))^c = (F, A)^c \vee (G, B)^c$.

Definition 10 ([8]). The union of two soft sets of (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$.

The following definition of the intersection of two soft sets is given as that of bi-intersection in [16].

Definition 11 ([16]). The intersection (H, C) of two soft sets (F, A) and (G, B) over a common universe U, denoted $(F, A) \cap (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

3. Soft topology and soft topological spaces

Let X be an initial universe set and E be the non-empty set of parameters.

Definition 12. The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \setminus (G, E)$, is defined as $H(e) = F(e) \setminus G(e)$ for all $e \in E$.

Definition 13. Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ read as x belongs to the soft set (F, E) whenever $x \in F(\alpha)$ for all $\alpha \in E$.

Note that for any $x \in X$, $x \notin (F, E)$, if $x \notin F(\alpha)$ for some $\alpha \in E$.

Definition 14. Let Y be a non-empty subset of X, then $\widetilde{Y}$ denotes the soft set (Y, E) over X for which $Y(\alpha) = Y$, for all $\alpha \in E$.

In particular, (X, E) will be denoted by $\widetilde{X}$.

Definition 15. Let $x \in X$, then (x, E) denotes the soft set over X for which $x(\alpha) = \{x\}$, for all $\alpha \in E$.

Definition 16. Let (F, E) be a soft set over X and Y be a non-empty subset of X. Then the sub soft set of (F, E) over Y denoted by $({}^Y F, E)$, is defined as follows

$${}^Y F(\alpha) = Y \cap F(\alpha), \quad \text{for all } \alpha \in E.$$

In other words $({}^Y F, E) = \widetilde{Y} \cap (F, E)$.

Definition 17. The *relative complement of a soft set* (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$ where $F' : A \to \mathcal{P}(U)$ is a mapping given by

$$F'(\alpha) = U - F(\alpha) \quad \text{for all } \alpha \in A.$$

Proposition 3. *Let* (F, E) *and* (G, E) *be the soft sets over* X. *Then*
(1) $((F, E) \cup (G, E))' = (F, E)' \cap (G, E)'$,
(2) $((F, E) \cap (G, E))' = (F, E)' \cup (G, E)'$.

Proof. (1) Let $(F, E) \cup (G, E) = (H, E)$ where, $H(e) = F(e) \cup G(e)$, for all $e \in E$. Then

$$\begin{aligned} H'(e) &= (F(e) \cup G(e))^c \\ &= (F(e))^c \cap (G(e))^c \\ &= (F^c(e)) \cap (G^c(e)) \quad \text{for all } e \in E. \end{aligned}$$

Thus $(H, E)' = (F, E)' \cap (G, E)'$, i.e. $((F, E) \cup (G, E))' = (F, E)' \cap (G, E)'$.

(2) Let $(F, E) \cap (G, E) = (H, E)$ where, $H(e) = F(e) \cap G(e)$, for all $e \in E$. Then

$$\begin{aligned} H'(e) &= (F(e) \cap G(e))^c \\ &= (F(e))^c \cup (G(e))^c \\ &= (F^c(e)) \cup (G^c(e)) \quad \text{for all } e \in E. \end{aligned}$$

Thus $(H, E)' = (F, E)' \cup (G, E)'$, i.e. $((F, E) \cap (G, E))' = (F, E)' \cup (G, E)'$. □

Definition 18. Let $\mathcal{T}$ be the collection of soft sets over X, then $\mathcal{T}$ is said to be a soft topology on X if

(1) $\Phi, \widetilde{X}$ belong to $\mathcal{T}$
(2) the union of any number of soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$
(3) the intersection of any two soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The triplet $(X, \mathcal{T}, E)$ is called a soft topological space over X.

Definition 19. Let $(X, \mathcal{T}, E)$ be a soft space over X, then the members of $\mathcal{T}$ are said to be soft open sets in X.

Definition 20. Let $(X, \mathcal{T}, E)$ be a soft space over X. A soft set (F, E) over X is said to be a soft closed set in X, if its relative complement $(F, E)'$ belongs to $\mathcal{T}$.

Proposition 4. *Let $(X, \mathcal{T}, E)$ be a soft space over X. Then*

(1) *$\Phi, \widetilde{X}$ are closed soft sets over X*
(2) *the intersection of any number of soft closed sets is a soft closed set over X*
(3) *the union of any two soft closed sets is a soft closed set over X.*

Proof. Follows from the definition of soft topological spaces and De-Morgan's laws for soft sets which are given in Proposition 3. □

Definition 21. Let X be an initial universe set, E be the set of parameters and $\mathcal{T} = \{\Phi, \widetilde{X}\}$. Then $\mathcal{T}$ is called the soft indiscrete topology on X and $(X, \mathcal{T}, E)$ is said to be a soft indiscrete space over X.

Definition 22. Let X be an initial universe set, E be the set of parameters and let $\mathcal{T}$ be the collection of all soft sets which can be defined over X. Then $\mathcal{T}$ is called the soft discrete topology on X and $(X, \mathcal{T}, E)$ is said to be a soft discrete space over X.

Proposition 5. *Let $(X, \mathcal{T}, E)$ be a soft space over X. Then the collection $\mathcal{T}_\alpha = \{F(\alpha) \mid (F, E) \in \mathcal{T}\}$ for each $\alpha \in E$, defines a topology on X.*

Proof. By definition, for any $\alpha \in E$, we have $\mathcal{T}_\alpha = \{F(\alpha) \mid (F, E) \in \mathcal{T}\}$. Now,

(1) $\Phi, \widetilde{X} \in \mathcal{T}$ implies that $\emptyset, X \in \mathcal{T}_\alpha$.
(2) Let $\{F_i(\alpha) \mid i \in I\}$ be a collection of sets in $\mathcal{T}_\alpha$. Since $(F_i, E) \in \mathcal{T}$, for all $i \in I$ so that $\cup_{i\in I}(F_i, E) \in \mathcal{T}$ thus $\cup_{i\in I} F_i(\alpha) \in \mathcal{T}_\alpha$.
(3) Let $F(\alpha), G(\alpha) \in \mathcal{T}_\alpha$ for some $(F, E), (G, E) \in \mathcal{T}$. Since $(F, E) \cap (G, E) \in \mathcal{T}$ so $F(\alpha) \cap G(\alpha) \in \mathcal{T}_\alpha$.

Thus $\mathcal{T}_\alpha$ defines a topology on X for each $\alpha \in E$. □

Proposition 5 shows that corresponding to each parameter $\alpha \in E$, we have a topology $\mathcal{T}_\alpha$ on X. Thus a soft topology on X gives a parameterized family of topologies on X.

Example 1. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ where $(F_1, E), (F_2, E), (F_3, E), (F_4, E)$ are soft sets over X, defined as follows

$$\begin{aligned} &F_1(e_1) = \{h_2\}, \qquad F_1(e_2) = \{h_1\}, \\ &F_2(e_1) = \{h_2, h_3\}, \qquad F_2(e_2) = \{h_1, h_2\}, \\ &F_3(e_1) = \{h_1, h_2\}, \qquad F_3(e_2) = X, \\ &F_4(e_1) = \{h_1, h_2\}, \qquad F_4(e_2) = \{h_1, h_3\}. \end{aligned}$$

Then $\mathcal{T}$ defines a soft topology on X and hence $(X, \mathcal{T}, E)$ is a soft topological space over X. It can be easily seen that

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_2\}, \{h_2, h_3\}, \{h_1, h_2\}\}$$

and

$$\mathcal{T}_{e_2} = \{\emptyset, X, \{h_1\}, \{h_1, h_3\}, \{h_1, h_2\}\}$$

are topologies on X.

Now we give an example to show that the converse of above proposition does not hold.

Example 2. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ where $(F_1, E), (F_2, E), (F_3, E), (F_4, E)$ are soft sets over X, defined as follows

$$F_1(e_1) = \{h_2\}, \qquad F_1(e_2) = \{h_1\},$$
$$F_2(e_1) = \{h_2, h_3\}, \qquad F_2(e_2) = \{h_1, h_2\},$$
$$F_3(e_1) = \{h_1, h_2\}, \qquad F_3(e_2) = \{h_1, h_2\},$$
$$F_4(e_1) = \{h_2\}, \qquad F_4(e_2) = \{h_1, h_3\}.$$

Then $\mathcal{T}$ is not a soft topology on X because $(F_2, E) \cup (F_3, E) = (G, E)$, where $G(e_1) = X$, and $G(e_2) = \{h_1, h_2\}$ and so $(G, E) \notin \mathcal{T}$.

Also,

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_2\}, \{h_2, h_3\}, \{h_1, h_2\}\}$$

and

$$\mathcal{T}_{e_2} = \{\emptyset, X, \{h_1\}, \{h_1, h_3\}, \{h_1, h_2\}\}$$

are topologies on X.

Hence Example 2 shows that any collection of soft sets need not to be a soft topology on X, even if the collection corresponding to each parameter defines a topology on X.

Proposition 6. *Let $(X, \mathcal{T}_1, E)$ and $(X, \mathcal{T}_2, E)$ be two soft topological spaces over the same universe X, then $(X, \mathcal{T}_1 \cap \mathcal{T}_2, E)$ is a soft topological space over X.*

Proof. (1) $\Phi, \widetilde{X}$ belong to $\mathcal{T}_1 \cap \mathcal{T}_2$.

(2) Let $\{(F_i, E) \mid i \in I\}$ be a family of soft sets in $\mathcal{T}_1 \cap \mathcal{T}_2$. Then $(F_i, E) \in \mathcal{T}_1$ and $(F_i, E) \in \mathcal{T}_2$, for all $i \in I$, so $\cup_{i\in I}(F_i, E) \in \mathcal{T}_1$ and $\cup_{i\in I}(F_i, E) \in \mathcal{T}_2$. Thus $\cup_{i\in I}(F_i, E) \in \mathcal{T}_1 \cap \mathcal{T}_2$.

(3) Let $(F, E), (G, E) \in \mathcal{T}_1 \cap \mathcal{T}_2$. Then $(F, E), (G, E) \in \mathcal{T}_1$ and $(F, E), (G, E) \in \mathcal{T}_2$. Since $(F, E) \cap (G, E) \in \mathcal{T}_1$ and $(F, E) \cap (G, E) \in \mathcal{T}_2$, so $(F, E) \cap (G, E) \in \mathcal{T}_1 \cap \mathcal{T}_2$.

Thus $\mathcal{T}_1 \cap \mathcal{T}_2$ defines a soft topology on X and $(X, \mathcal{T}_1 \cap \mathcal{T}_2, E)$ is a soft topological space over X. □

Remark 1. The union of two soft topologies on X may not be a soft topology on X.

Example 3. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mathcal{T}_1 = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ $\mathcal{T}_2 = \{\Phi, \widetilde{X}, (G_1, E), (G_2, E), (G_3, E), (G_4, E)\}$ be two soft topologies defined on X where $(F_1, E), (F_2, E), (F_3, E), (F_4, E), (G_1, E), (G_2, E), (G_3, E)$ and (G_4, E) are soft sets over X, defined as follows

$$F_1(e_1) = \{h_2\}, \qquad F_1(e_2) = \{h_1\},$$
$$F_2(e_1) = \{h_2, h_3\}, \qquad F_2(e_2) = \{h_1, h_2\},$$
$$F_3(e_1) = \{h_1, h_2\}, \qquad F_3(e_2) = X,$$
$$F_4(e_1) = \{h_1, h_2\}, \qquad F_4(e_2) = \{h_1, h_3\}$$

and

$$G_1(e_1) = \{h_2\}, \qquad G_1(e_2) = \{h_1\},$$
$$G_2(e_1) = \{h_2, h_3\}, \qquad G_2(e_2) = \{h_1, h_2\},$$
$$G_3(e_1) = \{h_1, h_2\}, \qquad G_3(e_2) = \{h_1, h_2\},$$
$$G_4(e_1) = \{h_2\}, \qquad G_4(e_2) = \{h_1, h_3\}.$$

Now, we define

$$\begin{aligned}\mathcal{T} &= \mathcal{T}_1 \cup \mathcal{T}_2 \\ &= \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (G_3, E), (G_4, E)\}.\end{aligned}$$

If we take

$$(F_2, E) \cup (G_3, E) = (H, E)$$

then

$$\begin{aligned}H(e_1) &= F_2(e_1) \cup G_3(e_1) \\ &= \{h_2, h_3\} \cup \{h_1, h_2\} \\ &= X\end{aligned}$$

and

$$\begin{aligned} H(e_2) &= F_2(e_2) \cup G_3(e_2) \\ &= \{h_1, h_2\} \cup \{h_1, h_2\} \\ &= \{h_1, h_2\} \end{aligned}$$

but $(H, E) \notin \mathcal{T}$. Thus $\mathcal{T}$ is not a soft topology on X.

Definition 23. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and (F, E) be a soft set over X. Then the soft closure of (F, E), denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E).

Clearly $\overline{(F, E)}$ is the smallest soft closed set over X which contains (F, E), by Proposition 4.

Theorem 1. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X, (F, E) and (G, E) are soft sets over X. Then*

(1) $\overline{\Phi} = \Phi$ *and* $\overline{\widetilde{X}} = \widetilde{X}$
(2) $(F, E) \widetilde{\subseteq} \overline{(F, E)}$
(3) (F, E) *is a soft closed set if and only if* $(F, E) = \overline{(F, E)}$
(4) $\overline{\overline{(F, E)}} = \overline{(F, E)}$
(5) $(F, E) \widetilde{\subseteq} (G, E)$ *implies* $\overline{(F, E)} \widetilde{\subseteq} \overline{(G, E)}$
(6) $\overline{(F, E) \cup (G, E)} = \overline{(F, E)} \cup \overline{(G, E)}$
(7) $\overline{(F, E) \cap (G, E)} \widetilde{\subseteq} \overline{(F, E)} \cap \overline{(G, E)}$.

Proof. (1) and (2) are obvious.

(3) If (F, E) is a soft closed set over X then (F, E) is itself a soft closed set over X which contains (F, E). So (F, E) is the smallest soft closed set containing (F, E) and $(F, E) = \overline{(F, E)}$.
Conversely suppose that $(F, E) = \overline{(F, E)}$. Since $\overline{(F, E)}$ is a soft closed set, so (F, E) is a soft closed set over X.
(4) Since $\overline{(F, E)}$ is a soft closed set therefore by part (3) we have $\overline{\overline{(F, E)}} = \overline{(F, E)}$.
(5) Suppose that $(F, E) \widetilde{\subseteq} (G, E)$. Then every soft closed super set of (G, E) will also contain (F, E). This means every soft closed super set of (G, E) is also a soft closed super set of (F, E). Hence the intersection of soft closed super sets of (F, E) is contained in the soft intersection of soft closed super sets of (G, E). Thus $\overline{(F, E)} \widetilde{\subseteq} \overline{(G, E)}$.
(6) Since $(F, E) \widetilde{\subseteq} (F, E) \cup (G, E)$ and $(G, E) \widetilde{\subseteq} (F, E) \cup (G, E)$. So by part (5), $\overline{(F, E)} \widetilde{\subseteq} \overline{(F, E) \cup (G, E)}$ and $\overline{(G, E)} \widetilde{\subseteq} \overline{(F, E) \cup (G, E)}$. Thus $\overline{(F, E)} \cup \overline{(G, E)} \widetilde{\subseteq} \overline{(F, E) \cup (G, E)}$. Conversely suppose that $(F, E) \widetilde{\subseteq} \overline{(F, E)}$ and $(G, E) \widetilde{\subseteq} \overline{(G, E)}$. So $(F, E) \cup (G, E) \widetilde{\subseteq} \overline{(F, E)} \cup \overline{(G, E)}$. By Proposition 5 $\overline{(F, E)} \cup \overline{(G, E)}$ is a soft closed set over X being the union of two soft closed sets. Then $\overline{(F, E) \cup (G, E)} \widetilde{\subseteq} \overline{(F, E)} \cup \overline{(G, E)}$. Thus $\overline{(F, E) \cup (G, E)} = \overline{(F, E)} \cup \overline{(G, E)}$.
(7) Since $(F, E) \cap (G, E) \widetilde{\subseteq} (F, E)$ and $(F, E) \cap (G, E) \widetilde{\subseteq} (G, E)$. So by part (5) $\overline{(F, E) \cap (G, E)} \widetilde{\subseteq} \overline{(F, E)}$ and $\overline{(F, E) \cap (G, E)} \widetilde{\subseteq} \overline{(G, E)}$. Thus $\overline{(F, E) \cap (G, E)} \widetilde{\subseteq} \overline{(F, E)} \cap \overline{(G, E)}$. □

Definition 24. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and (F, E) be a soft set over X. Then we associate with (F, E) a soft set over X, denoted by $(\overline{F}, E)$ and defined as
$\overline{F}(\alpha) = \overline{F(\alpha)}$, where $\overline{F(\alpha)}$ is the closure of $F(\alpha)$ in $\mathcal{T}_\alpha$ for each $\alpha \in E$.

Proposition 7. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and (F, E) be a soft set over X. Then $(\overline{F}, E) \widetilde{\subseteq} \overline{(F, E)}$.*

Proof. For any $\alpha \in E$, $\overline{F(\alpha)}$ is the smallest closed set in $(X, \mathcal{T}_\alpha)$ which contains $F(\alpha)$. Moreover if $\overline{(F, E)} = (H, E)$ then $H(\alpha)$ is also a closed set in $(X, \mathcal{T}_\alpha)$ containing $F(\alpha)$. This implies that $\overline{F}(\alpha) = \overline{F(\alpha)} \subseteq H(\alpha)$. Thus $(\overline{F}, E) \widetilde{\subseteq} \overline{(F, E)}$. □

Corollary 1. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and (F, E) be a soft set over X. Then $(\overline{F}, E) = \overline{(F, E)}$ if and only if $(\overline{F}, E)' \in \mathcal{T}$.*

Proof. If $(\overline{F}, E) = \overline{(F, E)}$ then $(\overline{F}, E)$ is a soft closed set and so $(\overline{F}, E)' \in \mathcal{T}$. Conversely if $(\overline{F}, E)' \in \mathcal{T}$ then $(\overline{F}, E)$ is a soft closed set containing (F, E). By Proposition 7 $(\overline{F}, E) \widetilde{\subseteq} \overline{(F, E)}$ and by the definition of soft closure of (F, E), any soft closed set over X which contains (F, E) will contain $\overline{(F, E)}$. Therefore $\overline{(F, E)} \widetilde{\subseteq} (\overline{F}, E)$. Thus $(\overline{F}, E) = \overline{(F, E)}$. □

Example 4. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), \ldots, (F_7, E)\}$ where

$$\begin{aligned} &F_1(e_1) = \{h_1, h_2\}, \quad F_1(e_2) = \{h_1, h_2\}, \\ &F_2(e_1) = \{h_2\}, \quad F_2(e_2) = \{h_1, h_3\}, \\ &F_3(e_1) = \{h_2, h_3\}, \quad F_3(e_2) = \{h_1\}, \\ &F_4(e_1) = \{h_2\}, \quad F_4(e_2) = \{h_1\}, \\ &F_5(e_1) = \{h_1, h_2\}, \quad F_5(e_2) = X, \\ &F_6(e_1) = X \quad F_6(e_2) = \{h_1, h_2\}, \\ &F_7(e_1) = \{h_2, h_3\}, \quad F_7(e_2) = \{h_1, h_3\}. \end{aligned}$$

Then $(X, \mathcal{T}, E)$ is a soft topological space over X.

Let (F, E) and (G, E) are defined as follows:

$$F(e_1) = \{h_1, h_3\}, \qquad F(e_2) = \emptyset,$$
$$G(e_1) = \{h_2, h_3\}, \qquad G(e_2) = \{h_1, h_2\}.$$

Then $(F, E) \cap (G, E) = ((F \cap G), E)$ is given by

$$(F \cap G)(e_1) = \{h_3\}, \qquad (F \cap G)(e_2) = \emptyset.$$

Now,

$$\overline{(F, E)} = \widetilde{X} \cap (F_2, E)' \cap (F_4, E)' = (F_2, E)',$$

and

$$\overline{(G, E)} = \widetilde{X}.$$

Therefore

$$\overline{(F, E)} \cap \overline{(G, E)} = \overline{(F, E)}.$$

Also

$$\begin{aligned} \overline{(F, E) \cap (G, E)} &= \cap\{\widetilde{X}, (F_1, E)', (F_2, E)', (F_4, E)', (F_5, E)'\} \\ &= (F_5, E)'. \end{aligned}$$

So

$$\overline{(F, E) \cap (G, E)} \widetilde{\subset} \overline{(F, E)} \cap \overline{(G, E)}$$

but

$$\overline{(F, E) \cap (G, E)} \neq \overline{(F, E)} \cap \overline{(G, E)}.$$

Next we see that

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_2\}, \{h_2, h_3\}, \{h_1, h_2\}\}$$

and

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_1\}, \{h_1, h_3\}, \{h_1, h_2\}\}.$$

Here $(\overline{F}, E)$ is given by

$$\overline{F}(e_1) = \{h_1, h_3\}, \qquad \overline{F}(e_2) = \emptyset.$$

Clearly

$$\overline{(F, E)} \widetilde{\subset} (\overline{F}, E) \quad \text{but} \quad \overline{(F, E)} \neq (\overline{F}, E).$$

Definition 25. Let $(X, \mathcal{T}, E)$ be a soft topological space over X, (G, E) be a soft set over X and $x \in X$. Then x is said to be a soft interior point of (G, E) if there exists a soft open set (F, E) such that $x \in (F, E) \widetilde{\subset} (G, E)$.

Definition 26. Let $(X, \mathcal{T}, E)$ be a soft topological space over X, (G, E) be a soft set over X and $x \in X$. Then (G, E) is said to be a soft neighborhood of x if there exists a soft open set (F, E) such that $x \in (F, E) \widetilde{\subset} (G, E)$.

Proposition 8. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X, (G, E) be a soft set over X and $x \in X$. If x is a soft interior point of (G, E) then x is an interior point of $G(\alpha)$ in $(X, \mathcal{T}_\alpha)$, for each $\alpha \in E$.*

Proof. For any $\alpha \in E$, $G(\alpha) \subseteq X$. If $x \in X$ is a soft interior point of (G, E) then there exists $(F, E) \in \mathcal{T}$ such that $x \in (F, E) \widetilde{\subset} (G, E)$. This means that, $x \in F(\alpha) \subseteq G(\alpha)$. As $F(\alpha) \in \mathcal{T}_\alpha$, so $F(\alpha)$ is an open set in $\mathcal{T}_\alpha$ and $x \in F(\alpha)$. This implies that x is an interior point of $G(\alpha)$ in $\mathcal{T}_\alpha$. $\square$

Proposition 9. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X, then*

(1) *each $x \in X$ has a soft neighborhood;*
(2) *if (F, E) and (G, E) are soft neighborhoods of some $x \in X$, then $(F, E) \cap (G, E)$ is also a soft neighborhood of x.*
(3) *if (F, E) is a soft neighborhood of $x \in X$ and $(F, E) \widetilde{\subset} (G, E)$, then (G, E) is also a soft neighborhood of $x \in X$.*

Proof. (1) For any $x \in X$, $x \in \widetilde{X}$ and since $\widetilde{X} \in \mathcal{T}$, so

$$x \in \widetilde{X} \widetilde{\subset} \widetilde{X}.$$

Thus $\widetilde{X}$ is a soft neighborhood of x.

(2) Let (F, E) and (G, E) be the soft neighborhoods of $x \in X$, then there exist $(F_1, E), (F_2, E) \in \mathcal{T}$ such that

$$x \in (F_1, E) \tilde{\subset} (F, E)$$

and

$$x \in (F_2, E) \tilde{\subset} (G, E).$$

Now $x \in (F_1, E)$ and $x \in (F_2, E)$ implies that $x \in (F_1, E) \cap (F_2, E)$ and $(F_1, E) \cap (F_2, E) \in \mathcal{T}$. So we have

$$x \in (F_1, E) \cap (F_2, E) \tilde{\subset} (F, E) \cap (G, E).$$

Thus $(F, E) \cap (G, E)$ is a soft neighborhood of x.

(3) Let (F, E) be a soft neighborhood of $x \in X$ and $(F, E) \tilde{\subset} (G, E)$. By definition there exists a soft open set (F_1, E) such that

$$x \in (F_1, E) \tilde{\subset} (F, E) \tilde{\subset} (G, E).$$

Thus

$$x \in (F_1, E) \tilde{\subset} (G, E).$$

Hence (G, E) is a soft neighborhood of x. □

Proposition 10. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X. For any soft open set (F, E) over X, (F, E) is a soft neighborhood of each point of $\cap_{\alpha \in E} F(\alpha)$.*

Proof. Let $(F, E) \in \mathcal{T}$. For any $x \in \cap_{\alpha \in E} F(\alpha)$, we have $x \in F(\alpha)$ for each $\alpha \in E$. Thus

$$x \in (F, E) \tilde{\subset} (F, E)$$

and so (F, E) is a soft neighborhood of x. □

Definition 27. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and Y be a non-empty subset of X. Then

$$\mathcal{T}_Y = \{({}^Y F, E) \mid (F, E) \in \mathcal{T}\}$$

is said to be the soft relative topology on Y and $(Y, \mathcal{T}_Y, E)$ is called a soft subspace of $(X, \mathcal{T}, E)$.

We can easily verify that $\mathcal{T}_Y$ is, in fact, a soft topology on Y.

Example 5. Any soft subspace of a soft discrete topological space is a soft discrete topological space.

Example 6. Any soft subspace of a soft indiscrete topological space is a soft indiscrete topological space.

Proposition 11. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and Y be a non-empty subset of X. Then $(Y, \mathcal{T}_{\alpha Y})$ is a subspace of $(X, \mathcal{T}_\alpha)$ for each $\alpha \in E$.*

Proof. Since $(Y, \mathcal{T}_Y, E)$ is a soft topological space over Y so $(Y, \mathcal{T}_{\alpha Y})$ is a topological space for each $\alpha \in E$. Now, by definition, for any $\alpha \in E$

$$\begin{aligned}\mathcal{T}_{\alpha Y} &= \{{}^Y F(\alpha) \mid (F, E) \in \mathcal{T}\} \\ &= \{Y \cap F(\alpha), \mid (F, E) \in \mathcal{T}\} \\ &= \{Y \cap F(\alpha), \mid F(\alpha) \in \mathcal{T}_\alpha\}.\end{aligned}$$

Thus $(Y, \mathcal{T}_{\alpha Y})$ is a subspace of $(X, \mathcal{T}_\alpha)$. □

Proposition 12. *Let $(Y, \mathcal{T}_Y, E)$ be a soft subspace of a soft topological space $(X, \mathcal{T}, E)$ and (F, E) be a soft open set in Y. If $\widetilde{Y} \in \mathcal{T}$ then $(F, E) \in \mathcal{T}$.*

Proof. Let (F, E) be a soft open set in Y, then there exists a soft open set (G, E) in X such that $(F, E) = \widetilde{Y} \cap (G, E)$. Now, if $\widetilde{Y} \in \mathcal{T}$ then $\widetilde{Y} \cap (G, E) \in \mathcal{T}$ by the third axiom of the definition of a soft topological space and hence $(F, E) \in \mathcal{T}$. □

Theorem 2. *Let $(Y, \mathcal{T}_Y, E)$ be a soft subspace of soft topological space $(X, \mathcal{T}, E)$ and (F, E) be a soft set over X, then*

(1) *(F, E) is soft open in Y if and only if $(F, E) = \widetilde{Y} \cap (G, E)$ for some $(G, E) \in \mathcal{T}$*

(2) *(F, E) is soft closed in Y if and only if $(F, E) = \widetilde{Y} \cap (G, E)$ for some soft closed set (G, E) in X.*

Proof. (1) Follows from the definition of a soft subspace.
(2) If (F, E) is soft closed in Y then we have

$$(F,E) = \widetilde{Y} \setminus (G,E), \quad \text{for some } (G,E) \in \mathcal{T}_Y.$$

Now, $(G,E) = \widetilde{Y} \cap (H,E)$, for some $(H,E) \in \mathcal{T}$.

For any $\alpha \in E$,

$$\begin{aligned} F(\alpha) &= Y(\alpha) \setminus G(\alpha) \\ &= Y \setminus G(\alpha) \\ &= Y \setminus (Y(\alpha) \cap H(\alpha)) \\ &= Y \setminus (Y \cap H(\alpha)) \\ &= Y \setminus H(\alpha) \\ &= Y \cap (X \setminus H(\alpha)) \\ &= Y \cap (H(\alpha))^c \\ &= Y(\alpha) \cap (H(\alpha))^c. \end{aligned}$$

Thus $(F,E) = \widetilde{Y} \cap (H,E)'$ where $(H,E)'$ is soft closed in X as $(H,E) \in \mathcal{T}$.
Conversely, assume that $(F,E) = \widetilde{Y} \cap (G,E)$ for some soft closed set (G,E) in X. This means that $(G,E)' \in \mathcal{T}$.
Now, if $(G,E) = (X,E) \setminus (H,E)$ where $(H,E) \in \mathcal{T}$ then for any $\alpha \in E$

$$\begin{aligned} F(\alpha) &= Y(\alpha) \cap G(\alpha) \\ &= Y \cap G(\alpha) \\ &= Y \cap (X(\alpha) \setminus H(\alpha)) \\ &= Y \cap (X \setminus H(\alpha)) \\ &= Y \setminus H(\alpha) \\ &= Y \setminus (Y \cap H(\alpha)) \\ &= Y(\alpha) \setminus (Y(\alpha) \cap H(\alpha)). \end{aligned}$$

Thus $(F,E) = \widetilde{Y} \setminus (\widetilde{Y} \cap (H,E))$. Since $(H,E) \in \mathcal{T}$, so
$(\widetilde{Y} \cap (H,E)) \in \mathcal{T}_Y$ and hence (F,E) is soft closed in Y. □

3.1. Soft separation axioms

Definition 28. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F,E) and (G,E) such that
$x \in (F,E)$ and $y \notin (F,E)$ or
$y \in (G,E)$ and $x \notin (G,E)$, then $(X, \mathcal{T}, E)$ is called a soft T_0-*space*.

Definition 29. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F,E) and (G,E) such that
$x \in (F,E)$ and $y \notin (F,E)$ and
$y \in (G,E)$ and $x \notin (G,E)$, then $(X, \mathcal{T}, E)$ is called a soft T_1-*space*.

Theorem 3. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X. If (x, E) is a soft closed set in $\mathcal{T}$ for each $x \in X$ then $(X, \mathcal{T}, E)$ is a soft T_1-space.*

Proof. Suppose that for each $x \in X (x, E)$ is a soft closed set in $\mathcal{T}$ then $(x,E)'$ is a soft open set in $\mathcal{T}$. Let $x, y \in X$ such that $x \neq y$. For $x \in X$, $(x,E)'$ is a soft open set such that $y \in (x,E)'$ and $x \notin (x,E)'$. Similarly $(y,E)' \in \mathcal{T}$ is such that $x \in (y,E)'$ and $y \notin (y,E)'$. Thus $(X, \mathcal{T}, E)$ is a soft T_1-*space* over X. □

Remark 2. The converse of Theorem 3 does not hold in general.

Example 7. Let $X = \{h_1, h_2\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1,E), (F_2,E), (F_3,E)\}$ where

$$\begin{aligned} &F_1(e_1) = X, && F_1(e_2) = \{h_2\}, \\ &F_2(e_1) = \{h_1\}, && F_2(e_2) = X, \\ &F_3(e_1) = \{h_1\}, && F_3(e_2) = \{h_2\}. \end{aligned}$$

Then $(X, \mathcal{T}, E)$ is a soft topological space over X. We have

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_1\}\}$$

and

$$\mathcal{T}_{e_2} = \{\emptyset, X, \{h_2\}\}.$$

Neither $(X, \mathcal{T}_{e_1})$ nor $(X, \mathcal{T}_{e_2})$ is a T_1-*space* but $h_1, h_2 \in X$ and $h_1 \neq h_2$, also

$$h_2 \in (F_1, E) \quad \text{but} \quad h_1 \notin (F_1, E)$$

and

$$h_1 \in (F_2, E) \quad \text{but} \quad h_2 \notin (F_2, E).$$

Thus $(X, \mathcal{T}, E)$ is a soft T_1-*space* over X.

We note that for (h_1, E), (h_2, E) over X, where

$$h_1(e_1) = \{h_1\}, \qquad h_1(e_2) = \{h_1\},$$
$$h_2(e_1) = \{h_2\}, \qquad h_2(e_2) = \{h_2\}.$$

The relative complement sets $(h_1, E)'$, $(h_2, E)'$ over X are defined by

$$h'(e_1) = \{h_2\}, \qquad h'(e_2) = \{h_2\},$$
$$h'(e_1) = \{h_1\}, \qquad h'(e_2) = \{h_1\}.$$

Neither $(h_1, E)'$ nor $(h_2, E)'$ belong to $\mathcal{T}$. This shows that the converse of the above theorem does not hold.

We also note that, in Example 5, $(X, \mathcal{T}, E)$ is a soft T_1-*space* but $(X, \mathcal{T}_\alpha)$ is not a T_1-*space* for every parameter $\alpha \in E$. The following two propositions give us the conditions to fix this problem.

Proposition 13. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F, E) and (G, E) such that $x \in (F, E)$ and $y \in (F, E)'$ or $y \in (G, E)$ and $x \in (G, E)'$, then $(X, \mathcal{T}, E)$ is a soft T_0-space and $(X, \mathcal{T}_\alpha)$ is a T_0-space, for each $\alpha \in E$.*

Proof. Let $x, y \in X$ such that $x \neq y$ and (F, E) and (G, E) are soft open sets over X such that $x \in (F, E)$ and $y \in (F, E)'$ or $y \in (G, E)$ and $x \in (G, E)'$. If $y \in (F, E)'$ then $y \in (F(\alpha))'$ for each $\alpha \in E$. This implies that $y \notin F(\alpha)$ for each $\alpha \in E$. Therefore $y \notin (F, E)$. Similarly we can show that if $x \in (G, E)'$ then $x \notin (G, E)$. Hence $(X, \mathcal{T}, E)$ is a soft T_0-*space*. Now for any $\alpha \in E$, $(X, \mathcal{T}_\alpha)$ is a topological space and $x \in (F, E)$ and $y \in (F, E)'$ Or $y \in (G, E)$ and $x \in (G, E)'$. So that $x \in F(\alpha)$ and $y \notin F(\alpha)$, or $y \in G(\alpha)$ and $x \notin G(\alpha)$. Thus $(X, \mathcal{T}_\alpha)$ is a T_0-*space*. □

Proposition 14. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F, E) and (G, E) such that $x \in (F, E)$ and $y \in (F, E)'$ And $y \in (G, E)$ and $x \in (G, E)'$, then $(X, \mathcal{T}, E)$ is a soft T_1-space and $(X, \mathcal{T}_\alpha)$ is a T_1-space, for each $\alpha \in E$.*

Proof. The proof is similar to the proof of Proposition 13. □

Proposition 15. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and Y be a non-empty subset of X. If $(X, \mathcal{T}, E)$ is a soft T_0-space then $(Y, \mathcal{T}_Y, E)$ is a soft T_0-space.*

Proof. Let $x, y \in Y$ such that $x \neq y$. Then there exist soft open sets (F, E) and (G, E) in X such that $x \in (F, E)$ and $y \notin (F, E)$ Or $y \in (G, E)$ and $x \notin (G, E)$. Now $x \in Y$ implies that $x \in \widetilde{Y}$. So $x \in \widetilde{Y}$ and $x \in (F, E)$.

Hence $x \in \widetilde{Y} \cap (F, E) = ({}^Y F, E)$ where $(F, E) \in \mathcal{T}$.

Consider $y \notin (F, E)$, this means that $y \notin F(\alpha)$ for some $\alpha \in E$. Then $y \notin Y \cap F(\alpha) = Y(\alpha) \cap F(\alpha)$. Therefore $y \notin \widetilde{Y} \cap (F, E) = ({}^Y F, E)$. Similarly it can be proved that if $y \in (G, E)$ and $x \notin (G, E)$ then $y \in ({}^Y G, E)$ and $x \notin ({}^Y G, E)$. Thus $(Y, \mathcal{T}_Y, E)$ is a soft T_0-*space*. □

Proposition 16. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and Y be a non-empty subset of X. If $(X, \mathcal{T}, E)$ is a soft T_1-space then $(Y, \mathcal{T}_Y, E)$ is a soft T_1-space.*

Proof. The proof is similar to the proof of Proposition 15. □

Definition 30. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F, E) and (G, E) such that

$x \in (F, E)$, $y \in (G, E)$ and $(F, E) \cap (G, E) = \Phi$, then $(X, \mathcal{T}, E)$ is called a soft T_2-*space*.

Proposition 17. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X. If $(X, \mathcal{T}, E)$ is a soft T_2-space over X then $(X, \mathcal{T}_\alpha)$ is a T_2-space for each $\alpha \in E$.*

Proof. Suppose that $(X, \mathcal{T}, E)$ is a soft T_2-*space* over X. For any $\alpha \in E$

$$\mathcal{T}_\alpha = \{F(\alpha) \mid (F, E) \in \mathcal{T}\}.$$

Let $x, y \in X$ such that $x \neq y$, then there exist (F, E), $(G, E) \in \mathcal{T}$ such that

$$x \in (F, E), y \in (G, E) \quad \text{and} \quad (F, E) \cap (G, E) = \Phi.$$

This implies that

$$x \in F(\alpha), y \in G(\alpha) \quad \text{and} \quad F(\alpha) \cap G(\alpha) = \emptyset.$$

Thus $(X, \mathcal{T}_\alpha)$ is a T_2-*space*, for each $\alpha \in E$. □

Remark 3. (1) Every soft T_1-*space* is a soft T_0-*space*.
(2) Every soft T_2-*space* is a soft T_1-*space*.

Proof. Let $(X, \mathcal{T}, E)$ be a soft topological space over X and $x, y \in X$ such that $x \neq y$.

(1) If $(X, \mathcal{T}, E)$ is a soft T_1-*space* then there exist soft open sets (F, E) and (G, E) such that $x \in (F, E)$ and $y \notin (F, E)$, and $y \in (G, E)$ and $x \notin (G, E)$. Obviously then we have $x \in (F, E)$ and $y \notin (F, E)$, or $y \in (G, E)$ and $x \notin (G, E)$. Thus $(X, \mathcal{T}, E)$ is a soft T_0-*space*.
(2) If $(X, \mathcal{T}, E)$ is a soft T_2-*space* then there exist soft open sets (F, E) and (G, E) such that

$$x \in (F, E), y \in (G, E) \quad \text{and} \quad (F, E) \cap (G, E) = \Phi.$$

Since $(F, E) \cap (G, E) = \Phi$, so $x \notin (G, E)$ and $y \notin (F, E)$. Hence $(X, \mathcal{T}, E)$ is a soft T_1-*space*. □

Example 8. Let $X = \{h_1, h_2\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where

$$F_1(e_1) = X, \qquad F_1(e_2) = \{h_2\},$$
$$F_2(e_1) = \{h_1\}, \qquad F_2(e_2) = X,$$
$$F_3(e_1) = \{h_1\}, \qquad F_3(e_2) = \{h_2\}.$$

Then $(X, \mathcal{T}, E)$ is a soft topological space over X. Also $(X, \mathcal{T}, E)$ is a soft T_1-*space* over X but not a soft T_2-*space* because $h_1, h_2 \in X$ and there do not exist any soft open sets (F, E) and (G, E) in X such that
$h_1 \in (F, E)$, $h_2 \in (G, E)$ and $(F, E) \cap (G, E) = \Phi$.
Now consider the following soft topology on X,
$\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E)\}$ where

$$F_1(e_1) = X, \qquad F_1(e_2) = \{h_2\}.$$

Then $(X, \mathcal{T}, E)$ is a soft topological space over X. Also $(X, \mathcal{T}, E)$ is a soft T_0-*space* over X which is not a soft T_1-*space* because $h_1, h_2 \in X$ but there do not exist soft open sets (F, E) and (G, E), such that
$h_1 \in (F, E)$ and $h_2 \notin (F, E)$ And $h_2 \in (G, E)$ and $h_1 \notin (G, E)$.

Proposition 18. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and Y be a non-empty subset of X. If $(X, \mathcal{T}, E)$ is a soft T_2-space then $(Y, \mathcal{T}_Y, E)$ is a soft T_2-space.*

Proof. Let $x, y \in Y$ such that $x \neq y$. Then there exist soft open sets (F, E) and (G, E) in X such that $x \in (F, E)$, $y \in (G, E)$ and $(F, E) \cap (G, E) = \Phi$.

So for each $\alpha \in E, x \in F(\alpha), y \in G(\alpha)$ and $F(\alpha) \cap G(\alpha) = \emptyset$. This implies that $x \in Y \cap F(\alpha), y \in Y \cap F(\alpha)$ and $F(\alpha) \cap G(\alpha) = \emptyset$. Hence $x \in ({}^YF, E) y \in ({}^YG, E)$ and $({}^YF, E) \cap ({}^YG, E) = \Phi$ where $({}^YF, E), ({}^YG, E) \in \mathcal{T}_Y$. Thus $(Y, \mathcal{T}_Y, E)$ is a soft T_2-*space*. □

Definition 31. Let $(X, \mathcal{T}, E)$ be a soft topological space over X, (G, E) be a soft closed set in X and $x \in X$ such that $x \notin (G, E)$. If there exist soft open sets (F_1, E) and (F_2, E) such that
$x \in (F_1, E)$, $(G, E) \widetilde{\subset} (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \Phi$, then $(X, \mathcal{T}, E)$ is called a soft regular space.

Definition 32. Let $(X, \mathcal{T}, E)$ be a soft topological space over X. Then $(X, \mathcal{T}, E)$ is said to be a soft T_3-*space* if it is soft regular and soft T_1-*space*.

Remark 4. (1) A soft T_3-*space* may not be a soft T_2-*space*.
(2) If $(X, \mathcal{T}, E)$ is a soft T_3-*space* then $(X, \mathcal{T}_\alpha)$ may not be a T_3-*space* for each parameter $\alpha \in E$.

Example 9. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), \ldots, (F_{30}, E)\}$ where

$$F_1(e_1) = X \qquad F_1(e_2) = \emptyset$$
$$F_2(e_1) = \{h_1\} \qquad F_2(e_2) = \emptyset$$
$$F_3(e_1) = \{h_2\} \qquad F_3(e_2) = \emptyset$$
$$F_4(e_1) = \{h_3\} \qquad F_4(e_2) = \emptyset$$
$$F_5(e_1) = \{h_1, h_2\} \qquad F_5(e_2) = \emptyset$$
$$F_6(e_1) = \{h_1, h_3\} \qquad F_6(e_2) = \emptyset$$

$F_7(e_1) = \{h_2, h_3\} \quad F_7(e_2) = \emptyset$
$F_8(e_1) = X \quad F_8(e_2) = \{h_1\}$
$F_9(e_1) = \{h_1\} \quad F_9(e_2) = \{h_1\}$
$F_{10}(e_1) = \{h_2\} \quad F_{10}(e_2) = \{h_1\}$
$F_{11}(e_1) = \{h_3\} \quad F_{11}(e_2) = \{h_1\}$
$F_{12}(e_1) = \{h_1, h_2\} \quad F_{12}(e_2) = \{h_1\}$
$F_{13}(e_1) = \{h_1, h_3\} \quad F_{13}(e_2) = \{h_1\}$
$F_{14}(e_1) = \{h_2, h_3\} \quad F_{14}(e_2) = \{h_1\}$
$F_{15}(e_1) = \emptyset \quad F_{15}(e_2) = \{h_1\}$
$F_{16}(e_1) = X \quad F_{16}(e_2) = \{h_2, h_3\}$
$F_{17}(e_1) = \{h_1\} \quad F_{17}(e_2) = \{h_2, h_3\}$
$F_{18}(e_1) = \{h_2\} \quad F_{18}(e_2) = \{h_2, h_3\}$
$F_{19}(e_1) = \{h_3\} \quad F_{19}(e_2) = \{h_2, h_3\}$
$F_{20}(e_1) = \{h_1, h_2\} \quad F_{20}(e_2) = \{h_2, h_3\}$
$F_{21}(e_1) = \{h_1, h_3\} \quad F_{21}(e_2) = \{h_2, h_3\}$
$F_{22}(e_1) = \{h_2, h_3\} \quad F_{22}(e_2) = \{h_2, h_3\}$
$F_{23}(e_1) = \emptyset \quad F_{23}(e_2) = \{h_2, h_3\}$
$F_{24}(e_1) = \{h_1\} \quad F_{24}(e_2) = X$
$F_{25}(e_1) = \{h_2\} \quad F_{25}(e_2) = X$
$F_{26}(e_1) = \{h_3\} \quad F_{26}(e_2) = X$
$F_{27}(e_1) = \{h_1, h_2\} \quad F_{27}(e_2) = X$
$F_{28}(e_1) = \{h_1, h_3\} \quad F_{28}(e_2) = X$
$F_{29}(e_1) = \{h_2, h_3\} \quad F_{29}(e_2) = X$
$F_{30}(e_1) = \emptyset \quad F_{30}(e_2) = X.$

Then $(X, \mathcal{T}, E)$ is a soft topological space over X. We note that $(X, \mathcal{T}, E)$ is a soft T_3-*space* but it is not a soft T_2-*space* because $h_2, h_3 \in X$ but there do not exist soft open sets (F, E) and (G, E) such that
$h_2 \in (F, E)$, $h_3 \in (G, E)$ and $(F, E) \cap (G, E) = \Phi$. Thus every soft T_3-*space* is not necessarily a soft T_2-*space*.
Now we have,

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}\}$$

and

$$\mathcal{T}_{e_2} = \{\emptyset, X, \{h_1\}, \{h_2, h_3\}\}.$$

Here $(X, \mathcal{T}_{e_2})$ is not a T_3-*space*. This shows that if $(X, \mathcal{T}, E)$ is a soft T_3-*space* then $(X, \mathcal{T}_\alpha)$ need not to be a T_3-*space* for each parameter $\alpha \in E$.

Proposition 19. *Let $(X, \mathcal{T}, E)$ be a soft topological space over X and Y be a non-empty subset of X. If $(X, \mathcal{T}, E)$ is a soft T_3-space then $(Y, \mathcal{T}_Y, E)$ is a soft T_3-space.*

Proof. By Proposition 17, $(Y, \mathcal{T}_Y, E)$ is a soft T_1-*space*. Let $y \in Y$ and (F, E) be a soft closed set in Y such that $y \notin (F, E)$. Then $y \notin ((Y, E) \cap (G, E))$ where $(F, E) = ((Y, E) \cap (G, E))$ for some soft closed set in X, by Theorem 2. But $y \in (Y, E)$, so $y \notin (G, E)$. As $(X, \mathcal{T}, E)$ is a soft T_3-*space*, so there exist soft open sets (G_1, E) and (G_2, E) in X such that
$y \in (G_1, E)$, $(G, E)\tilde{\subset}(G_2, E)$ and $(G_1, E) \cap (G_2, E) = \Phi$. Now if we take

$$(F_1, E) = (Y, E) \cap (G_1, E) \quad \text{and} \quad (F_2, E) = (Y, E) \cap (G_2, E),$$

then (F_1, E) and $(F_2, E) \in \mathcal{T}_Y$ such that $y \in (F_1, E)$ and $(F, E)\tilde{\subset}(Y, E) \cap (G_2, E) = (F_2, E)$ and $(F_1, E) \cap (F_2, E)\tilde{\subset}(G_1, E) \cap (G_2, E) = \Phi$, i.e. $(F_1, E) \cap (F_2, E) = \Phi$. Thus $(Y, \mathcal{T}_Y, E)$ is a soft T_3-*space*. □

Definition 33. Let $(X, \mathcal{T}, E)$ be a soft topological space over X, (F, E) and (G, E) soft closed sets over X such that $(F, E) \cap (G, E) = \Phi$. If there exist soft open sets (F_1, E) and (F_2, E) such that
$(F, E)\tilde{\subset}(F_1, E)$, $(G, E)\tilde{\subset}(F_2, E)$ and $(F_1, E) \cap (F_2, E) = \Phi$,
then $(X, \mathcal{T}, E)$ is called a soft normal space.

Definition 34. Let $(X, \mathcal{T}, E)$ be a soft topological space over X. Then $(X, \mathcal{T}, E)$ is said to be a soft T_4-*space* if it is soft normal and soft T_1-*space*.

Remark 5. (1) A soft T_4-*space* need not be a soft T_3-*space*.
(2) If $(X, \mathcal{T}, E)$ is a soft T_4-*space* then $(X, \mathcal{T}_\alpha)$ need not be a T_4-*space* for each parameter $\alpha \in E$.
(3) If $(X, \mathcal{T}, E)$ is a soft T_4-*space* and Y is a non-empty subset of X then $(Y, \mathcal{T}_Y, E)$ need not be a soft T_4-*space*.

Example 10. Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1, e_2\}$ and $\mathcal{T} = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), \ldots, (F_8, E)\}$ where

$$F_1(e_1) = \{h_1, h_2, h_4\}, \qquad F_1(e_2) = \{h_1, h_2, h_3\},$$
$$F_2(e_1) = \{h_1, h_3, h_4\}, \qquad F_2(e_2) = \{h_1, h_2, h_3\},$$
$$F_3(e_1) = \{h_1, h_4\}, \qquad F_3(e_2) = \{h_1, h_2, h_3\},$$
$$F_4(e_1) = \{h_2, h_3\}, \qquad F_4(e_2) = \{h_1, h_2, h_3\},$$
$$F_5(e_1) = \{h_2\}, \qquad F_5(e_2) = \{h_1, h_2, h_3\},$$
$$F_6(e_1) = \{h_3\}, \qquad F_6(e_2) = \{h_1, h_2, h_3\},$$
$$F_7(e_1) = \emptyset, \qquad F_7(e_2) = \{h_1, h_2, h_3\},$$
$$F_8(e_1) = X, \qquad F_8(e_2) = \{h_1, h_2, h_3\}.$$

Then $(X, \mathcal{T}, E)$ is a soft topological space over X. We note that $(X, \mathcal{T}, E)$ is a soft T_4-*space* but it is not a soft T_3-*space* because $h_1 \in X$ and $(F_3, E)'$ is a soft closed set in X such that $h_1 \notin (F_3, E)'$ but there do not exist soft open sets (F, E) and (G, E) such that $h_1 \in (F, E)$, $(F_3, E)' \widetilde{\subset} (G, E)$ and $(F, E) \cap (G, E) = \Phi$. Thus every soft T_4-*space* is not necessarily a soft T_3-*space*.

Now we have,

$$\mathcal{T}_{e_1} = \{\emptyset, X, \{h_1, h_2, h_4\}, \{h_1, h_3, h_4\}, \{h_1, h_4\}, \{h_2, h_3\}, \{h_2\}, \{h_3\}\}$$

and

$$\mathcal{T}_{e_2} = \{\emptyset, X, \{h_1, h_2, h_3\}\}.$$

Here $(X, \mathcal{T}_{e_1})$ and $(X, \mathcal{T}_{e_2})$ are not T_3-*spaces*. This shows that if $(X, \mathcal{T}, E)$ is a soft T_4-*space* then $(X, \mathcal{T}_\alpha)$ need not be a T_4-*space* for each parameter $\alpha \in E$.

We take

$$Y = \{h_1, h_2, h_3\}.$$

Then

$$\mathcal{T}_Y = \{\Phi, \widetilde{Y}, ({}^YF_1, E), ({}^YF_2, E), ({}^YF_3, E), \ldots, ({}^YF_8, E)\}$$

where

$${}^YF_1(e_1) = \{h_1, h_2\}, \qquad {}^YF_1(e_2) = Y,$$
$${}^YF_2(e_1) = \{h_1, h_3\}, \qquad {}^YF_2(e_2) = Y,$$
$${}^YF_3(e_1) = \{h_1\}, \qquad {}^YF_3(e_2) = Y,$$
$${}^YF_4(e_1) = \{h_2, h_3\}, \qquad {}^YF_4(e_2) = Y,$$
$${}^YF_5(e_1) = \{h_2\}, \qquad {}^YF_5(e_2) = Y,$$
$${}^YF_6(e_1) = \{h_3\}, \qquad {}^YF_6(e_2) = Y,$$
$${}^YF_7(e_1) = \emptyset, \qquad {}^YF_7(e_2) = Y,$$
$${}^YF_8(e_1) = Y \qquad {}^YF_8(e_2) = Y.$$

We note that $(Y, \mathcal{T}_Y, E)$ is not a soft T_4-*space* because $({}^YF_3, E)'$ and $({}^YF_4, E)'$ are soft closed sets in Y such that $({}^YF_3, E)' \cap ({}^YF_4, E)' = \Phi$ but there do not exist any soft open sets (F, E) and (G, E) in Y such that $({}^YF_3, E)' \widetilde{\subset} (F, E)$, $(F_4, E)' \widetilde{\subset} (G, E)$ and $(F, E) \cap (G, E) = \Phi$. Thus a soft subspace of a soft T_4-*space* may not be a soft T_4-*space*.

Conclusion 1. *We have introduced soft topological spaces which are defined over an initial universe with a fixed set of parameters. The notions of soft open sets, soft closed sets, soft closure, soft interior points, soft neighborhood of a point and soft separation axioms are introduced and their basic properties are investigated. In the end, we must say that, this paper is just a beginning of a new structure and we have studied a few ideas only, it will be necessary to carry out more theoretical research to establish a general framework for the practical application.*

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